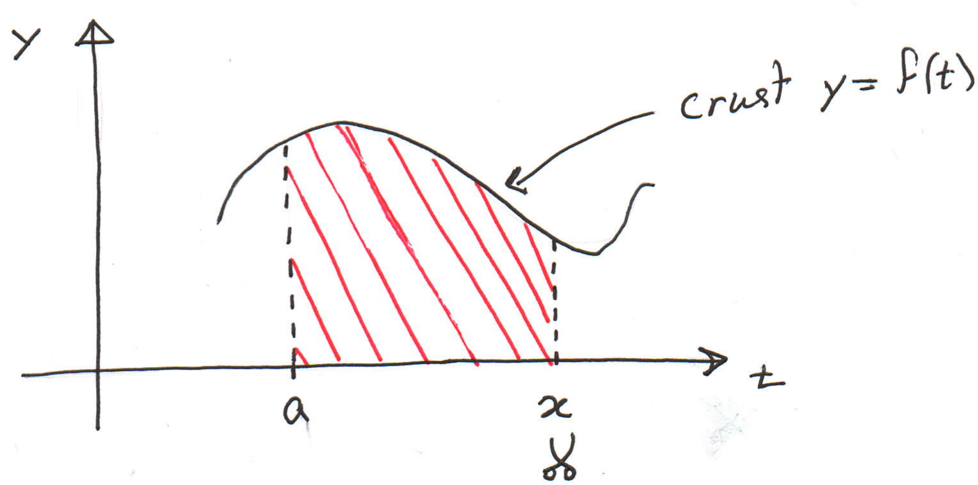
The Fundamental theorem of Calculus

Lecture 6

The fundamental theorem of calculus is the engine that makes the integration shortcut work. You have heard about it many times in statements about the conservation of momentum and energy.
In probability theory, this theorem lurks behind considerations of the cumulative distribution function, making cumulative distributions very useful tools in the study of continuous random variables.
In its naked form, the fundamental theorem of calculus is a statement on the existance of anti-derivatives for continuous functions. For example, we can guess that $\int \cos x \, dx = \sin x + C$, but what about $\int \cos(x^2) dx$ or $\int e^{-x^2} dx$? Do these functions have anti-derivatives at all?
As you may have already guessed, anti-derivatives and areas under curves are related.

Def: Given a continuous curve $y = f(x)$, we will call $F(x) = \int_a^x f(t)\,dt$ the "slice of cake" function.

Story: Suppose your favorite aunt, aunt Jamaina, bakes you a cake (Because you're in Calc. I this cake is two dimensional, but with a little bit of practice you can learn to eat 12 dimensional cake. Just yesterday one was baked from 12 students). The crust of the cake is the curve $y = f(t)$. She makes a cut at the point $x = a$ and asks you where else to cut to slice out a piece of cake. $F(x) = \int_a^x f(t)\,dt$ is the amount of cake you'll be getting if second slice is made at x.



Comprehension Check: Suppose $y = f(t)$ is continuous and non negative. In simple, non mathematical language, describe what you're telling the aunt if $F(x) = \int_a^x f(t)\,dt$ and

(a) $x = a$

(b) $x = b < a$.

Solution:

(a) F\$!?#.... you Aunt Jamaima! or Forgive me, but I am not hungry and want 0 cake. Thank You Aunt Jamaima!

(b) $F(b) = \int_a^b f(t)\,dt = -\int_b^a f(t)\,dt.$ or

I have eaten too much! Could you please remove some pounds off of me?

Ex. Describe the precise relationship between the function $f(x)$ and $F(x)$ where

(a) $f(x) = x$; $F(x) = \int_0^x t\,dt$

(b) $f(x) = x^2$; $F(x) = \int_0^x t^2 dt$

(c) $f(x) = x^3$; $F(x) = \int_0^x t^3 dt$

(d) $f(x) = 5x - x^2$; $F(x) = \int_1^x (5t - t^2) dt$

Solution:

(a) $$F(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(k \frac{x}{n}\right) \frac{x}{n} =$$

$$= \lim_{n \to \infty} \left(\frac{x}{n}\right)^2 \sum_{k=1}^{n} k = \lim_{n \to \infty} \left(\frac{x}{n}\right)^2 \frac{n(n+1)}{2} = \frac{1}{2} x^2$$

$F(x)$ is an anti-derivative of $f(x)$!

(b) $$F(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(k \frac{x}{n}\right)^2 \frac{x}{n} = \lim_{n \to \infty} \left(\frac{x}{n}\right)^3 \sum_{k=1}^{n} k^2$$

$$= \lim_{n \to \infty} \left(\frac{x}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} = \frac{2}{6} \cdot x^3 = \frac{1}{3} x^3$$

$F(x)$ is an anti-derivative of $f(x)$!

(c) $$F(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(k \frac{x}{n}\right)^3 \frac{x}{n} =$$

$$= \lim_{n \to \infty} \left(\frac{x}{n}\right)^4 \sum_{k=1}^{n} k^3 = \lim_{n \to \infty} \left(\frac{x}{n}\right)^4 \left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{4} x^4$$

$F(x)$ is an anti-derivative of $f(x)$!

(d) $F(x) = \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left[5\left(1 + k\frac{x-1}{n}\right) - \left(1 + k\frac{x-1}{n}\right)^2 \right] \frac{x-1}{n}$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left[5 + k \cdot 5 \frac{x-1}{n} - 1 - k \cdot 2 \frac{x-1}{n} - k^2 \left(\frac{x-1}{n}\right)^2 \right] \frac{x-1}{n}$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left[4 + k \cdot 3 \frac{x-1}{n} - k^2 \left(\frac{x-1}{n}\right)^2 \right] \frac{x-1}{n}$$

$$= \lim_{n \to \infty} \left(\frac{x-1}{n} \sum_{k=1}^{n} 4 + 3\left(\frac{x-1}{n}\right)^2 \sum_{k=1}^{n} k - \left(\frac{x-1}{n}\right)^3 \sum_{k=1}^{n} k^2 \right)$$

$$= \lim_{n \to \infty} \left(\frac{x-1}{n} \cdot 4n + 3\left(\frac{x-1}{n}\right)^2 \cdot \frac{n(n+1)}{2} - \left(\frac{x-1}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \right)$$

$$= 4(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$$

Notice that $F'(x) = 4 + 3(x-1) - (x-1)^2 =$

$= 5x - x^2$

Hence $F(x)$ is also an anti-derivative of $f(x)$.

This is not a coincidence!

Thm: (The Fundamental Theorem of Calculus)

Let $f(x)$ be continuous. Then $F(x) = \int_a^x f(t)\,dt$ is an anti-derivative of $f(x)$. That is $F'(x) = f(x)$.

Proof: By definition, $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left(F(x+h) - F(x)\right) = \lim_{h \to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

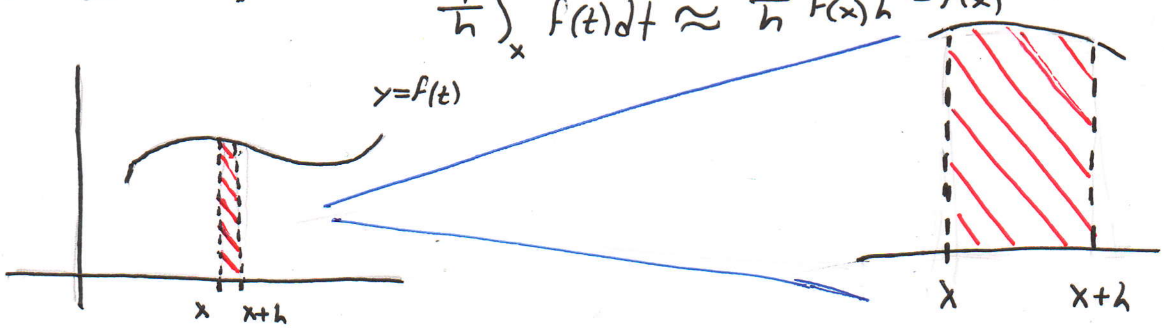
$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f_{ave}[x, x+h] =$$

$= f(x)$ because $f(t) \approx f(x)$ for all $x \leq t \leq x+h$ when h is small.

Thus $F'(x) = f(x)$ as desired.

The idea behind this theorem is clever, but once you catch on to it, it becomes very simple to see why it works:

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \frac{1}{h} f(x) h = f(x)$$



In other words, if f is continuous $\int_x^{x+h} f(t)\,dt$ is essentially the area of a rectangle of height $f(x)$ and width h.

When a simple anti-derivative formula can be guessed, the Fundamental Theorem of calculus gives rise to the integration shortcut:

Ex. Compute $\int_0^3 x^2\,dx$.

Solution:

Define $F(x) = \int_0^x t^2\,dt$. Then $\int_0^3 x^2\,dx =$

$= \int_0^3 t^2\,dt = F(3)$.

By FTC $F'(x) = x^2$ and this means $F(x) = \frac{1}{3}x^3 + C$.

(Notice how we used the idea that all anti-derivatives are geometrically the same curve!).

Now $0 = F(0) = \frac{1}{3}\cdot 0^3 + C = C$

Thus $F(x) = \frac{1}{3}x^3$ and $F(3) = \frac{1}{3}\cdot 3^3 = 9$.

In general if $f(x)$ is continuous and we can guess a simple anti-derivative $F(x)$ (i.e. $F'(x) = f(x)$) then $\int_a^b f(x)\,dx$ can be solved as follows:

Define $F_A(x) = \int_a^x f(t)\,dt$. Then by FTC

$F_A'(x) = f(x)$. Because anti-derivatives are essentially unique, it follows that $F_A(x) = F(x) + C$ for some constant C. Notice that $0 = F_A(a) = F(a) + C$.

Thus $C = -F(a)$.

In particular, $F_A(x) = F(x) - F(a)$.

Now $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = F_A(b) =$

$= F(b) - F(a) = F(x)\Big|_a^b$.

<u>Ex.</u> Solve

(a) $\int \cos x\,dx$

(b) $\int e^{-x^2}\,dx$

(c) $\int \tan^{-1}\left(\frac{v+1}{v^3}\right)dv$

(d) $\int u e^{\cos u}\,du$.

Solution:

(a) $\int \cos x \, dx = \sin x + C$. That's the answer that most calculus students give and it is, of course correct.

However, I had a student once that wrote

$$\int \cos x \, dx = \int_0^x \cos t \, dt + C.$$

There were many questions like this on the test and the student scored 100 without even so much as to strain himself.

Clearly the student was correct!

Ever since then I try to word questions differently.

(b) No simple formula can be obtained!

$$\int e^{-x^2} dx = \int_0^x e^{-t^2} dt + C$$

(c) Similarly, $\int \tan^{-1}\left(\frac{v+1}{v^3}\right) dv = \int_1^v \tan^{-1}\left(\frac{t+1}{t^3}\right) dt + C.$

(d) $\int u e^{\cos u} du = \int_0^u t e^{\cos t} dt + C.$

Ex. Find the derivative

(a) $\frac{d}{dx}\int_1^{x^4} \sec t\, dt$

(b) $\frac{d}{dx}\int_1^{x} \frac{1}{t^3+1}\, dt$

(c) $\frac{d}{dx}\int_x^{\pi} \sqrt{1+\sec t}\, dt$

(d) $\frac{d}{dx}\int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

(e) $\frac{d}{dx}\int_{\cos x}^{\sin x} \ln(1+2v)\, dv$

(f) $\frac{d}{dx}\int_x^{x^3} e^{-t^2}\, dt$

(g) $\frac{d}{dx}\int_0^{\int_1^x \cos(t^2)\,dt} e^{\tan^{-1} v}\, dv$

Solution:

(a) Let $F(x) = \int_1^x \sec t\, dt$

Then by chain rule $\frac{d}{dx}\int_1^{x^4} \sec t\, dt = \frac{d}{dx} F(x^4)$

$= F'(x^4)\, 4x^3 = \sec(x^4)\cdot 4x^3$

(b) $\frac{d}{dx}\int_1^{x} \frac{1}{t^3+1}\, dt = \frac{1}{x^3+1}$

(c) $\frac{d}{dx}\int_x^{\pi}\sqrt{1+\sec t}\,dt = -\frac{d}{dx}\int_{\pi}^{x}\sqrt{1+\sec t}\,dt$

$= -\sqrt{1+\sec x}$

(d) $\frac{d}{dx}\int_{\sin x}^{1}\sqrt{1+t^2}\,dt = -\frac{d}{dx}\int_{1}^{\sin x}\sqrt{1+t^2}\,dt$

$= -\sqrt{1+\sin^2 x}\ \cos x$

(e) $\frac{d}{dx}\int_{\cos x}^{\sin x}\ln(1+2v)\,dv =$

$= \frac{d}{dx}\left(\int_{\cos x}^{c}\ln(1+2v)\,dv + \int_{c}^{\sin x}\ln(1+2v)\,dv\right)$

$= \frac{d}{dx}\left(\int_{c}^{\sin x}\ln(1+2v)\,dv - \int_{c}^{\cos x}\ln(1+2v)\,dv\right)$

$= \ln(1+2\sin x)\cos x + \ln(1+2\cos x)\sin x$

In general $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)$.

(f) $\frac{d}{dx}\int_{x}^{x^3} e^{-t^2}\,dt = e^{-x^6}\cdot 3x^2 - e^{-x^2}$.

(g) By chain rule

$$\frac{d}{dx}\int_0^{\int_1^x \cos(t^2)\,dt} e^{\tan^{-1}(v)}\,dv = e^{\tan^{-1}\left(\int_1^x \cos t^2\,dt\right)} \cdot \cos x^2.$$

why do we change parameters?

Q. why do we write $F(x) = \int_a^x f(t)\,dt$ instead of $\int_a^x f(x)\,dx$?

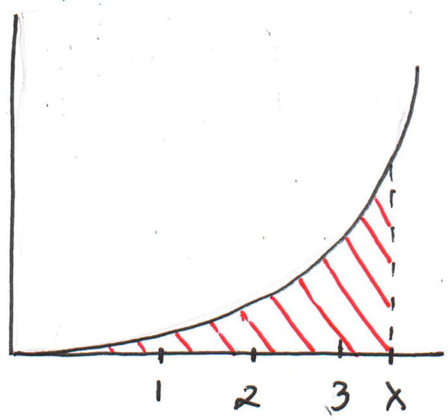
A. This is a matter of semantics. We wish to comunicate "taking area under the curve". Only the bounds of integration must change!

For example, lets compere $F(x) = \int_0^x t^2\,dt$ and $G(x) = \int_0^x x^2\,dx$.

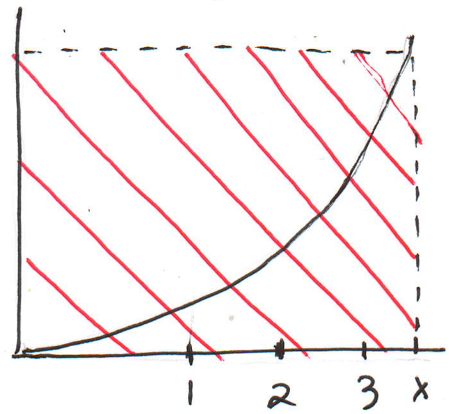
First notice that $F(1), F(2), F(3)$ look like $\int_0^1 t^2\,dt$, $\int_0^2 t^2\,dt$, $\int_0^3 t^2\,dt$ whereas

$G(1)$, $G(2)$, $G(3)$ look like

$$\int_0^1 1^2\,dx, \quad \int_0^2 2^2\,dx, \quad \int_0^3 3^2\,dx$$



$F(x)$



$G(x)$

in particular, whereas $F(x) = \frac{1}{3}x^3$, $G(x) = x^3$.